

ON THE RESIDUAL FINITENESS AND OTHER PROPERTIES OF (RELATIVE) ONE-RELATOR GROUPS

Stephen J. Pride

ABSTRACT. A relative one-relator presentation has the form $\mathcal{P} = \langle \mathbf{x}, H; R \rangle$ where \mathbf{x} is a set, H is a group, and R is a word on $\mathbf{x}^{\pm 1} \cup H$. We show that if the word on $\mathbf{x}^{\pm 1}$ obtained from R by deleting all the terms from H has what we call the *unique max-min property*, then the group defined by \mathcal{P} is residually finite if and only if H is residually finite (Theorem 1). We apply this to obtain new results concerning the residual finiteness of (ordinary) one-relator groups (Theorem 4). We also obtain results concerning the conjugacy problem for one-relator groups (Theorem 5), and results concerning the relative asphericity of presentations of the form \mathcal{P} (Theorem 6).

2000 Mathematics Subject Classification. Primary 20E26, 20F05; Secondary 20F10, 57M07.

Key words and phrases. Residual finiteness, one-relator group, relative presentation, (power) conjugacy problem, asphericity, unique max-min property, 2-complex of groups, covering complex.

1 Introduction

The question of when one-relator groups are residually finite is still open.

In the torsion-free case there are well-known examples of groups which are not residually finite, namely the Baumslag-Solitar/Meskin groups [4], [15]:

$$G = \langle \mathbf{x}; U^{-1}V^lUV^m \rangle,$$

where U, V do not generate a cyclic subgroup of the free group on \mathbf{x} , and $|l| \neq |m|$, $|l|, |m| > 1$. On the other hand, there are some examples which are known to be residually finite. For instance, it was shown in [3] that if

$$W = UV^{-1}, \tag{1}$$

where U, V are positive words on an alphabet \mathbf{x} and the exponent sum of x in UV^{-1} is 0 for each $x \in \mathbf{x}$, or if

$$W = [U, V], \tag{2}$$

where U, V are (not necessarily positive) words on \mathbf{x} such that no letter $x \in \mathbf{x}$ appears in both U and V , then $G = \langle \mathbf{x}; W \rangle$ is residually finite.

In the torsion case there is the well-known open question:

Question 1 [2], [5, Question OR1] Is every one-relator group with torsion residually finite?

Question 1 is known to be true when $G = \langle \mathbf{x}; W^n \rangle$ where W is a *positive* word and $n > 1$ [9] (see also [19]). In [20], Wise obtains further related results, summed up by his “Quasi-Theorem 1.3”: *If W is sufficiently positive, and W^n is sufficiently small cancellation, then G is residually finite.*

A related open question is:

Question 2 [5, Question OR6], [11, Question 8.68] If a torsion-free one-relator group $G_1 = \langle \mathbf{x}; W \rangle$ is residually finite, then is $G_n = \langle \mathbf{x}; W^n \rangle$ also residually finite for $n > 1$?

(Of course, if Question 1 is true, then Question 2 is trivially true.)

It was shown in [1] that Question 2 holds true when W has the form (1) or (2).

Here, amongst other things, we tackle Question 2 by considering *relative* presentations.

A relative presentation has the form

$$\mathcal{P} = \langle \mathbf{x}, H; \mathbf{r} \rangle$$

where H is a group and \mathbf{r} is a set of expressions of the form

$$R = x_1^{\varepsilon_1} h_1 x_2^{\varepsilon_2} h_2 \dots x_r^{\varepsilon_r} h_r \quad (r > 0, x_i \in \mathbf{x}, \varepsilon_i = \pm 1, h_i \in H, 1 \leq i \leq r). \quad (3)$$

The word

$$W = x_1^{\varepsilon_1} x_2^{\varepsilon_2} \dots x_r^{\varepsilon_r} \quad (r > 0, x_i \in \mathbf{x}, \varepsilon_i = \pm 1, 1 \leq i \leq r) \quad (4)$$

is called the *\mathbf{x} -skeleton* of R . We *do not* require that the \mathbf{x} -skeleton is reduced or cyclically reduced. The group $G = G(\mathcal{P})$ defined by \mathcal{P} is the quotient of $H * F$ (where F is the free group on \mathbf{x}) by the normal closure of the elements of $H * F$ represented by the expressions $R \in \mathbf{r}$. The composition of the canonical imbedding $H \rightarrow H * F$ with the quotient map $H * F \rightarrow G$ is called the *natural homomorphism*, denoted by $\nu : H \rightarrow G$ (or simply $H \rightarrow G$).

As is normal, we will often abuse notation and write $G = \langle \mathbf{x}, H; \mathbf{r} \rangle$, or $G \cong \langle \mathbf{x}, H; \mathbf{r} \rangle$.

When \mathbf{r} consists of a single element R , then we have the *one-relator relative presentation*

$$\mathcal{P} = \langle \mathbf{x}, H; R \rangle. \quad (5)$$

Heuristically, $G = G(\mathcal{P})$ should be governed by the “shape” of the \mathbf{x} -skeleton of R and the algebraic properties of H .

Here we introduce the *unique max-min property* for the “shape” of W . (Words of the form (1) are a very special case.) For a group H , denote by \mathcal{M}_H the class of one-relator relative presentations of the form (5), where W has the unique max-min property.

Theorem 1 *If \mathcal{P} is in \mathcal{M}_H then:*

- (i) *the natural homomorphism $H \rightarrow G(\mathcal{P})$ is injective;*
- (ii) *$G(\mathcal{P})$ is residually finite if and only if H is residually finite.*

We can deduce from this

Theorem 2 (Substitution Theorem). *Let K be a one-relator group given by an ordinary presentation $\langle \mathbf{y}, z; S(\mathbf{y}, z) \rangle$, and let $\mathcal{P} = \langle \mathbf{x}, H; R \rangle$ be an \mathcal{M}_H -presentation. Then the*

group given by the relative presentation $\langle \mathbf{x}, \mathbf{y}, H; S(\mathbf{y}, R) \rangle$ is residually finite if and only if H and K are residually finite.

We can give the proof of this straightaway. Consider the \mathcal{M}_{H*K} -presentation $\overline{\mathcal{P}} = \langle \mathbf{x}, H * K; Rz^{-1} \rangle$. By Theorem 1, $L = G(\overline{\mathcal{P}})$ is residually finite if and only if $H * K$ is residually finite, which is equivalent to requiring that both H and K are residually finite (using results discussed in [12] p417). Now note that

$$L \cong \langle \mathbf{x}, \mathbf{y}, z, H; S(\mathbf{y}, z), Rz^{-1} \rangle \cong \langle \mathbf{x}, \mathbf{y}, H; S(\mathbf{y}, R) \rangle.$$

In particular, taking K to be defined by $\langle z; z^n \rangle$ ($n > 1$) we have:

Theorem 3. *If $G = \langle \mathbf{x}, H; R \rangle$ is a residually finite \mathcal{M}_H -group, then the group $G_n = \langle \mathbf{x}, H; R^n \rangle$ ($n > 1$) is also residually finite.*

Now take H to be a free group Φ . Then \mathcal{M}_Φ -groups are one-relator groups. Since Φ is residually finite ([12], p116 or p417), we obtain the following theorem concerning residual finiteness of one-relator groups.

Theorem 4 *Every \mathcal{M}_Φ -group $G = \langle \mathbf{x}, \Phi; R \rangle$ is a residually finite one-relator group. Moreover, if $K = \langle \mathbf{y}, z; S(\mathbf{y}, z) \rangle$ is a one-relator group, then the one-relator group $\overline{K} = \langle \mathbf{x}, \mathbf{y}, \Phi; S(\mathbf{y}, R) \rangle$ is residually finite if and only if K is residually finite. In particular, $G_n = \langle \mathbf{x}, \Phi; R^n \rangle$ ($n > 1$) is residually finite.*

The solution of the conjugacy problem for one-relator groups with *torsion* has been solved by B.B.Newman [16]. However, for the *torsion-free* case the problem is still open [5, Question O5].

Theorem 5 *Every \mathcal{M}_Φ -group (Φ a finitely generated free group) has solvable conjugacy problem. Also, such groups have solvable power conjugacy problem.*

(Two elements c, d of a group are said to be *power conjugate* if some power of c is conjugate to some power of d .)

Other aspects of relative presentations (and in particular, one-relator relative presentations) have been studied intensively, particularly *asphericity*. Recall [6] that a relative presentation \mathcal{P} is *aspherical* (more accurately, *diagrammatically aspherical*) if every spherical picture over \mathcal{P} contains a dipole. Under a weaker condition on “shape” (the *unique min property*, or equivalently the *unique max property*) we can prove:

Theorem 6 *Let \mathcal{P} be a relative presentation as in (5), where W has the unique min property. Then \mathcal{P} is aspherical.*

It then follows from [6] (see Corollary 1 of Theorem 1.1, Theorem 1.3, and Theorem 1.4) that for the group $G = G(\mathcal{P})$ we have:

- (i) the natural homomorphism $H \rightarrow G$ is injective;
- (ii) every finite subgroup of G is contained in a conjugate of H ;
- (iii) for any left $\mathbb{Z}G$ -module A , and any right $\mathbb{Z}G$ -module B ,

$$H^n(G, A) \cong H^n(H, A),$$

$$H_n(G, B) \cong H_n(H, B)$$

for all $n \geq 3$.

2 Max-min property

Let \mathbf{x} be an alphabet. A *weight function* on \mathbf{x} is a function

$$\theta : \mathbf{x} \longrightarrow \mathbb{Z}$$

such that $\text{Im } \theta$ generates the additive group \mathbb{Z} (that is, $\gcd\{\theta(x) : x \in \mathbf{x}\}$ is 1). A *strict* weight function is one for which $\theta(x) \neq 0$ for all $x \in \mathbf{x}$.

Let W be a word on \mathbf{x} as in (4). Given a weight function θ , we then have the function

$$\phi = \phi_W^\theta : \{0, 1, 2, \dots, r\} \rightarrow \mathbb{Z},$$

$$\phi(j) = \sum_{i=0}^j \varepsilon_i \theta(x_i)$$

(where $\phi(0) = 0$ since the empty sum is taken to be 0). We will say that the weight function is *admissible* for W if $\phi(r) = 0$.

For visual purposes, it is useful to extend ϕ to a piecewise linear function $\phi : [0, r] \rightarrow \mathbb{R}$, so that the graph of ϕ in the interval $[j-1, j]$ is the straight line segment joining the points $(j-1, \phi(j-1))$, $(j, \phi(j))$ ($0 < j \leq r$). We will informally refer to this graph as “the graph of W ” (with respect to θ).

A word W as in (4) will be said to have the *unique max-min property* if for some admissible strict weight function θ , the graph of W has a unique maximum and a unique minimum. To be precise, we require that, for some admissible strict weight function, and some $k, l \in \{1, 2, \dots, r\}$, we have $\phi(j) < \phi(k)$ for all $j \in \{1, 2, \dots, r\} - k$ and $\phi(j) > \phi(l)$ for all $j \in \{1, 2, \dots, r\} - l$. We also require that $x_k \neq x_{k+1}$ and $x_l \neq x_{l+1}$ (subscripts modulo r). This amounts to requiring that W is “reduced at the unique maximum and minimum”, that is, $x_k^{\varepsilon_k} \neq x_{k+1}^{-\varepsilon_{k+1}}$, $x_l^{\varepsilon_l} \neq x_{l+1}^{-\varepsilon_{l+1}}$ (subscripts modulo r). For at the maximum and minimum we must have *either* $x_j \neq x_{j+1}$, *or* $x_j = x_{j+1}$ and $\varepsilon_j = -\varepsilon_{j+1}$ ($j = k, l$). If the two letters occurring at the unique maximum are not disjoint from the two letters occurring at the unique minimum (i.e. $\{x_k, x_{k+1}\} \cap \{x_l, x_{l+1}\}$ is not empty), then we will say that W has the *strong* unique max-min property.

A word W as in (4) will be said to have the *unique min property* if for some strict weight function θ , the graph of W has a unique minimum (but not necessarily a unique

maximum). The *unique max property* is defined similarly, but is not really of interest because replacing θ by $-\theta$ will convert this property to the unique min property.

We let \mathcal{M}_H^1 (respectively \mathcal{S}_H^1) denote the subclass of \mathcal{M}_H consisting of relative presentations of the form (5) for which W has the unique max-min property (respectively, the strong unique max-min property) with respect to the weight function

$$\mathbf{1} : \mathbf{x} \longrightarrow \mathbb{Z} \quad x \mapsto 1 \ (x \in \mathbf{x}).$$

Lemma 1 *Every \mathcal{M}_H -group can be embedded into an \mathcal{M}_H^1 -group.*

Proof. Let $G = \langle \mathbf{x}, H; R \rangle$ with R as in (3), and suppose $W = x_1^{\varepsilon_1} x_2^{\varepsilon_2} \dots x_r^{\varepsilon_r}$ has the unique max-min property with respect to some strict weight function $\theta : \mathbf{x} \rightarrow \mathbb{Z}$. We can assume $\theta(x) > 0$ for all x . For if $\theta(x) < 0$ then we can replace x by x^{-1} .

Let

$$\mathbf{y} = \{y : y \in \mathbf{x}, \theta(y) > 1\},$$

and let

$$\hat{\mathbf{x}} = (\mathbf{x} - \mathbf{y}) \cup \{y_1, y_2, \dots, y_{\theta(y)} : y \in \mathbf{y}\}.$$

Let $\hat{G} = \langle \hat{\mathbf{x}}, H; \hat{R} \rangle$, where \hat{R} is obtained from R by replacing each occurrence of $y^{\pm 1}$ by $(y_1 y_2 \dots y_{\theta(y)})^{\pm 1}$ ($y \in \mathbf{y}$). It is easy to see that the $\hat{\mathbf{x}}$ -skeleton \hat{W} of \hat{R} has the unique max-min property with respect to $\mathbf{1} : \hat{\mathbf{x}} \rightarrow \mathbb{Z}$. (The graph of \hat{W} is obtained from that of W by “stretching” along the horizontal axis.) Moreover, G is embedded into \hat{G} , for we have the retraction ρ with section μ :

$$\hat{G} \xrightleftharpoons[\mu]{\rho} G \quad \rho\mu = \text{id}_G$$

$$\begin{aligned} \rho : x &\mapsto x \ (x \in \mathbf{x} - \mathbf{y}), \ y_1 \mapsto y, y_i \mapsto 1 \ (y \in \mathbf{y}, 1 < i \leq \theta(y)), \ h \mapsto h \ (h \in H), \\ \mu : x &\mapsto x \ (x \in \mathbf{x} - \mathbf{y}), y \mapsto y_1 y_2 \dots y_{\theta(y)} \ (y \in \mathbf{y}), h \mapsto h \ (h \in H). \end{aligned}$$

Lemma 2 *Every \mathcal{M}_H^1 -group can be embedded into an \mathcal{S}_H^1 -group.*

Proof. Let $G = \langle \mathbf{x}, H; R \rangle$, where the \mathbf{x} -skeleton W of R has the unique max-min property with respect to the constant function $\mathbf{1} : \mathbf{x} \rightarrow \mathbb{Z}$. Suppose the letters occurring at the unique maximum are a, b , and those occurring at the unique minimum are c, d . We can assume that $\{a, b\} \cap \{c, d\}$ is empty, otherwise there is nothing to prove.

Let $\mathbf{y} = \mathbf{x} - \{a, b, c, d\}$, and introduce a new alphabet

$$\hat{\mathbf{x}} = \{a, b, c, d, e\} \cup \{y_1, y_2 : y \in \mathbf{y}\}.$$

Let \hat{R} be obtained from R as follows. For each $y \in \mathbf{y}$, replace all occurrences of $y^{\pm 1}$ by $(y_1 y_2)^{\pm 1}$, and replace all occurrences of $a^{\pm 1}$ (respectively, $b^{\pm 1}$, $c^{\pm 1}$, $d^{\pm 1}$) by $(ea)^{\pm 1}$ (respectively, $(be)^{\pm 1}$, $(ec)^{\pm 1}$, $(de)^{\pm 1}$). Let $\hat{G} = \langle \hat{\mathbf{x}}, H; \hat{R} \rangle$, and let \hat{W} be the word obtained from \hat{R} by deleting all terms from H . The graph of \hat{W} under the weight function $\mathbf{1} : \hat{\mathbf{x}} \rightarrow \mathbb{Z}$ is the graph of W magnified by a factor of 2, and e occurs at the unique maximum and

the unique minimum. Moreover, G is embedded into \hat{G} for we have the retraction ρ with section μ :

$$\hat{G} \xrightleftharpoons[\mu]{\rho} G \quad \rho\mu = \text{id}_G$$

$$\begin{aligned} \rho : z &\mapsto z \ (z \in \{a, b, c, d\}), e \mapsto 1, y_1 \mapsto y, y_2 \mapsto 1 \ (y \in \mathbf{y}), h \mapsto h \ (h \in H), \\ \mu : a &\mapsto ea, b \mapsto be, c \mapsto ec, d \mapsto de, y \mapsto y_1 y_2 \ (y \in \mathbf{y}), h \mapsto h \ (h \in H). \end{aligned}$$

Remark 1 Note that in both the above proofs we have $\mu\nu = \hat{\nu}$, where $\nu : H \rightarrow G$, $\hat{\nu} : H \rightarrow \hat{G}$ are the natural homomorphisms. Thus if $\hat{\nu}$ is injective then so is ν .

Remark 2 Note also from the proof of the above two lemmas we get that every \mathcal{M}_H -group is a retract of an \mathcal{S}_H^1 -group.

Remark 3 The referee has brought my attention to the work of K.S.Brown [8], which is concerned with whether a homomorphism χ from a one-relator group $B = \langle \mathbf{x}; W \rangle$ ($|\mathbf{x}| \geq 2$, W as in (4) and cyclically reduced) onto \mathbb{Z} has finitely generated kernel. Such a homomorphism is induced by a weight function θ which is admissible for W . However, since θ need not be strict, it is necessary to interpret the max-min property more widely. Thus the unique maximum could be a “plateau”: ie, for some $k \in \{1, 2, \dots, r\}$ we could have $\phi(k) = \phi(k+1)$ and $\phi(j) < \phi(k)$ for all $j \in \{1, 2, \dots, r\} - \{k, k+1\}$ (subscripts modulo r). Similarly, the unique minimum could be a “reverse plateau”. Then according to Brown [8], as restated in Theorem 2.2 of [13], $\ker \chi$ is finitely generated if and only if $|\mathbf{x}| = 2$, and W has the unique max-min property in the above sense with respect to the corresponding weight function. In our work we could also allow non-strict weight functions. However, for the most part this can be avoided. For example, if the unique maximum is a plateau with $x_k \neq x_{k+2}$ then we could transform it to a genuine maximum by deleting x_{k+1} from \mathbf{x} and replacing H by $H * \langle x_{k+1} \rangle$. However, if the unique maximum is a plateau with $x_k = x_{k+2}$ then some of our arguments need to be modified, which we leave as an exercise for the reader.

3 A construction

By a *2-complex of groups* we mean a connected graph of groups (in the sense of Serre [18]) with trivial edge groups, together with a set of closed paths, which we call *defining paths*. (These are essentially the “generalized complexes” defined in §1 of [10], where more details can be found. Note however, that in [10] a “2-cell” $c(\alpha)$ consists of *all* cyclic permutations of $\alpha^{\pm 1}$ for each one of our defining paths α . We specifically *do not* add these extra paths. This makes no significant difference.)

Let \mathcal{P} be as in (5), and let θ be an admissible weight function for W . There is then an induced epimorphism

$$\psi : G \rightarrow \mathbb{Z} \quad x \mapsto \theta(x) \ (x \in \mathbf{x}), h \mapsto 0 \ (h \in H).$$

We can construct a 2-complex of groups

$$\tilde{\mathcal{P}} = \langle \Gamma, H_n \ (n \in \mathbb{Z}); \ (n, R) \ (n \in \mathbb{Z}) \rangle$$

whose fundamental group is isomorphic to the kernel K of ψ . The underlying graph Γ has vertex set \mathbb{Z} , edges (n, x^ε) ($n \in \mathbb{Z}, x \in \mathbf{x}, \varepsilon = \pm 1$), and initial, terminal and inversion functions $\iota, \tau, ^{-1}$ given by $\iota(n, x^\varepsilon) = n, \tau(n, x^\varepsilon) = n + \varepsilon\theta(x), (n, x^\varepsilon)^{-1} = (n + \varepsilon\theta(x), x^{-\varepsilon})$. The vertex groups are copies $H_n = \{(n, h) : h \in H\}$ of H (with the obvious multiplication $(n, h)(n, h') = (n, hh')$). We extend $\iota, \tau, ^{-1}$ to the elements of the vertex groups by defining $\iota(n, h) = n = \tau(n, h), (n, h)^{-1} = (n, h^{-1})$ (where h^{-1} is the inverse of h in H). We extend θ to $\mathbf{x}^{\pm 1} \cup H$ by defining $\theta(x^{-1}) = -\theta(x)$ ($x \in \mathbf{x}$), $\theta(h) = 0$ ($h \in H$). Then for any sequence $\alpha = z_1 z_2 \dots z_q$ with $z_i \in \mathbf{x}^{\pm 1} \cup H$ and any vertex $n \in \Gamma$, we have a path (n, α) in the graph of groups starting at n , where

$$(n, \alpha) = (n, z_1)(n + \theta(z_1), z_2)(n + \theta(z_1) + \theta(z_2), z_3) \dots (n + \theta(z_1) + \theta(z_2) + \dots + \theta(z_{q-1}), z_q).$$

In particular we have the (closed) paths (n, R) .

There is an obvious action of \mathbb{Z} on the above graph of groups, with $i \in \mathbb{Z}$ acting on vertices by $i \cdot n = i + n$ ($n \in \mathbb{Z}$), and on the edges and vertex groups by $i.(n, z) = (i + n, z)$ ($n \in \mathbb{Z}, z \in \mathbf{x}^{\pm 1} \cup H$). This action of course extends to paths. Thus $(i, \alpha) = i.(0, \alpha)$. In particular, $(i, R) = i.(0, R)$, so \mathbb{Z} acts on $\tilde{\mathcal{P}}$.

If we regard \mathcal{P} as a 2-complex of groups with a single vertex o , edges x^ε ($x \in \mathbf{x}, \varepsilon = \pm 1$), vertex group H , and defining path R , then we have a mapping of 2-complexes of groups

$$\rho : \tilde{\mathcal{P}} \longrightarrow \mathcal{P}$$

$$n \mapsto o, \ (n, x^\varepsilon) \mapsto x^\varepsilon, \ (n, h) \mapsto h, \ (n, R) \mapsto R$$

($n \in \mathbb{Z}, x \in \mathbf{x}, \varepsilon = \pm 1, h \in H$). This induces a homomorphism

$$\rho_* : \pi_1(\tilde{\mathcal{P}}, 0) \longrightarrow \pi_1(\mathcal{P}, o) = G$$

which is injective, and $\text{Im} \rho_* = K$. This can easily be proved by adapting the standard arguments of covering space theory for ordinary 2-complexes (see for example [17] pp 157-159), to this relative situation.

4 Proof of Theorem 1

Since residual finiteness is closed under taking subgroups, it follows from Lemmas 1 and 2 and the Remark 1 at the end of §2 that it suffices to prove Theorem 1 for \mathcal{S}_H^1 -groups.

We will make use of the following results: (a) *A free product $F * B$, where F is a free group, is residually finite if and only if B is residually finite*; (b) *An infinite cyclic extension of a finitely generated group L is residually finite if and only if L is residually finite*. (The first of these follows from results on p417 of [12]; the second is a special case of Theorem 7, p29 of [14].)

We can assume \mathbf{x} is finite. For if not let \mathbf{x}' be the set of letters occurring in R . Then G is isomorphic to $G' * \Psi$ where $G' \cong \langle \mathbf{x}', H; R \rangle$, and Ψ is the free group on $\mathbf{x} - \mathbf{x}'$. So by (a) above, it is enough to work with G' .

Let G be defined by an \mathcal{S}_H^1 presentation as in (5), with $e \in \mathbf{x}$ occurring at both the unique maximum and the unique minimum of the graph of W under the weight function

$\theta = 1$. We denote the maximum and minimum values of ϕ_W by M, m respectively. Note that $m \leq 0 \leq M$ and $m < M$.

We first deal with the trivial case when $M - m = 1$. Then up to cyclic permutation and inversion, $R = eha^{-1}h'$, where $a \in \mathbf{x} - \{e\}$, $h, h' \in H$. Thus $G = \Phi * H$, where Φ is the free group on $\mathbf{x} - \{e\}$, so the theorem holds by (a) above.

Now suppose $M - m > 1$. Let $f \in \mathbf{x} - \{e\}$.

We have the epimorphism

$$\psi : G \rightarrow \mathbb{Z} \quad x \mapsto 1 \ (x \in \mathbf{x}), h \mapsto 0 \ (h \in H).$$

Also, we have the homomorphism

$$\eta : \mathbb{Z} \rightarrow G \quad 1 \mapsto f.$$

Then $\psi\eta = \text{id}_{\mathbb{Z}}$, so G is a semidirect product $K \rtimes \mathbb{Z}$, where $K = \ker \psi$, and with the action of $n \in \mathbb{Z}$ on K being induced by conjugation by f^n .

The fundamental group of $\tilde{\mathcal{P}}$ (at the vertex 0), as in §3, is isomorphic to K .

We will obtain a relative presentation for K by collapsing a maximal tree.

The edges $(n, f)^{\pm 1}$ form a maximal tree T in Γ . Let R_n be the word on $\{(i, x) : i \in \mathbb{Z}, x \in \mathbf{x}, x \neq f\} \cup (\bigcup_{i \in \mathbb{Z}} H_i)$ obtained from (n, R) by deleting all edges from T which occur in (n, R) and replacing all terms (i, x^{-1}) by $(i - 1, x)^{-1}$ ($i \in \mathbb{Z}, x \in \mathbf{x}, x \neq f$). Then

$$\mathcal{Q} = \langle (n, x) \ (n \in \mathbb{Z}, x \in \mathbf{x}, x \neq f), *_{n \in \mathbb{Z}} H_n; R_n \ (n \in \mathbb{Z}) \rangle$$

is a relative presentation for K . Moreover, since the edges in T constitute an orbit under the action of \mathbb{Z} on our graph of groups, the action of \mathbb{Z} on K is given by the automorphism

$$\mu : (n, x) \mapsto (n + 1, x) \ (x \in \mathbf{x}, x \neq f), \ (n, h) \mapsto (n + 1, h) \ (h \in H)$$

($n \in \mathbb{Z}$).

Now consider the HNN -extension \overline{K} of K given by the relative presentation

$$\begin{aligned} \overline{\mathcal{Q}} = & \langle (n, x) \ (n \in \mathbb{Z}, x \in \mathbf{x}, x \neq f), *_{n \in \mathbb{Z}} H_n, s; R_n \ (n \in \mathbb{Z}) \\ & s(n, x)s^{-1} = (n + 1, x) \ (n \in \mathbb{Z}, x \in \mathbf{x}, x \neq e, f), \\ & s(n, h)s^{-1} = (n + 1, h) \ (n \in \mathbb{Z}, h \in H) \rangle. \end{aligned}$$

The automorphism μ of K can be extended to an automorphism $\overline{\mu}$ of \overline{K} by defining $\overline{\mu}(s) = s$. Then $G = K \rtimes_{\mu} \mathbb{Z}$ can be embedded into $\overline{G} = \overline{K} \rtimes_{\overline{\mu}} \mathbb{Z}$.

By our assumption, up to cyclic permutation and inversion, $(0, R)$ will have the form

$$(M - 1, e)(M, h)(M - 1, a)^{-1}\gamma_0((m, b)^{-1}(m, h')(m, e))^{\varepsilon}\delta_0,$$

where $h, h' \in H, \varepsilon = \pm 1, a, b \in \mathbf{x} - \{e\}$, and each term (i, z) occurring in the paths γ_0, δ_0 is such that both its initial and terminal vertices lie in the range $m + 1, m + 2, \dots, M - 1$.

Then

$$R_0 = (M - 1, e)\alpha_0(m, e)^{\varepsilon}\beta_0$$

where α_0, β_0 do not contain any occurrence of $(i, e)^{\pm 1}$ with $i \leq m$ or $i \geq M - 1$. More generally, for $n \in \mathbb{Z}$

$$R_n = (n + M - 1, e)\alpha_n(n + m, e)^\varepsilon\beta_n$$

where α_n, β_n do not contain any occurrence of $(i, e)^{\pm 1}$ with $i \leq n + m$ or $i \geq n + M - 1$.

Let F_0 be the free group on

$$(\mathbf{x} - \{e, f\}) \cup \{s, (m + 1, e), (m + 2, e) \dots, (M - 1, e)\}.$$

Then there is a homomorphism

$$\overline{K} \rightarrow H * F_0$$

defined as follows:

$$\begin{aligned} s &\mapsto s, \\ (n, x) &\mapsto s^n x s^{-n} \quad (x \in \mathbf{x}, x \neq e, f, n \in \mathbb{Z}), \\ (n, h) &\mapsto s^n h s^{-n} \quad (h \in H, n \in \mathbb{Z}), \\ (i, e) &\mapsto (i, e) \quad (m + 1 \leq i \leq M - 1), \end{aligned}$$

and (inductively), for $k = 0, 1, 2, \dots$

$$(k + M, e) \mapsto \beta_{k+1}^{-1}(k + 1 + m, e)^{-\varepsilon}\alpha_{k+1}^{-1},$$

$$(-k + m, e) \mapsto (\beta_{-k}(-k + M - 1, e)\alpha_{-k})^{-\varepsilon}.$$

This homomorphism is actually an isomorphism. The inverse is defined by

$$\begin{aligned} x &\mapsto (0, x) \quad (x \in \mathbf{x}, x \neq e, f), \\ h &\mapsto (0, h) \quad (h \in H), \\ (i, e) &\mapsto (i, e) \quad m + 1 \leq i \leq M - 1, \\ s &\mapsto s. \end{aligned}$$

Thus \overline{G} is an infinite cyclic extension of the group $F_0 * H$.

Remark 4 Note that by sending s to the generator $1 \in \mathbb{Z} \subset G = K \rtimes_\mu \mathbb{Z}$, we obtain a retraction of \overline{G} onto G (with section induced by the inclusion of K into \overline{K}).

We can now complete the proof.

Clearly the natural homomorphism from H into \overline{G} is injective (and is thus injective into G). Hence if H is not residually finite then neither is G . It remains to show that if H is residually finite then so is \overline{G} (and thus G).

Case 1. If H is finitely generated then the result holds straight away by (a) and (b) above.

Case 2. Suppose that H is not finitely generated. For any homomorphism θ from H to a group H_θ we obtain an induced homomorphism from $\overline{G} = (F_0 * H) \rtimes_{\overline{\mu}} \mathbb{Z}$ to $\overline{G}_\theta = (F_0 * H_\theta) \rtimes_{\overline{\mu}} \mathbb{Z}$ which acts as θ on H and acts as the identity on F_0 and \mathbb{Z} .

Let $g = (w_0 h_1 w_1 \dots h_q w_q).n$ be a non-trivial element of \overline{G} (where $q \geq 0, h_1 \dots h_q \in H - \{1\}, w_1, \dots, w_{q-1} \in F_0 - \{1\}, w_0, w_q \in F_0, n \in \mathbb{Z}$, and if q is 0 then either $n \neq 0$ or w_0 is non-trivial). Since residually finite groups are fully residually finite, there is a homomorphism τ from H onto a finite group H_τ such that $\tau(h_i) \neq 1$ ($i = 1, \dots, q$). So the image of g in $\overline{G}_\tau = (F_0 * H_\tau) \rtimes_{\overline{\mu}} \mathbb{Z}$ is non-trivial, and then Case 1 applies.

5 Proof of Theorem 5

Lemma 3 *Let C be a group which is a retract of a group B . If B has solvable conjugacy (or power conjugacy) problem, then so does C .*

Proof. By assumption we have maps $B \xrightleftharpoons[\mu]{\rho} C, \rho\mu = \text{id}_C$. Clearly if $c, d \in C$ are conjugate (respectively, power conjugate) in C then $\mu(c), \mu(d)$ are conjugate (respectively, power conjugate) in B . Conversely if there exists $b \in B$ such that $b\mu(c)b^{-1} = \mu(d)$ (respectively, $b\mu(c)^i b^{-1} = \mu(d)^j$), then $\rho(b)c\rho(b)^{-1} = d$ (respectively, $\rho(b)c^i \rho(b)^{-1} = d^j$). Thus the result follows.

Now it is shown in [7] that infinite cyclic extensions of finitely generated free groups have solvable conjugacy, and power conjugacy, problem. By Remarks 2, 4, every \mathcal{M}_Φ -group is a retract of such a group.

6 Proof of Theorem 6

We will assume familiarity with the terminology in §§1.2, 1.4 of [6].

As in Lemma 1, we can assume that $\theta(x) > 0$ for all x . We can extend θ to any word $U = y_1^{\varepsilon_1} y_2^{\varepsilon_2} \dots y_s^{\varepsilon_s}$, ($s > 0, y_i \in \mathbf{x}, \varepsilon_i = \pm 1, 1 \leq i \leq s$) by $\theta(U) = \sum_{i=1}^s \varepsilon_i \theta(y_i)$.

Let \mathbb{P} be a based connected spherical picture (with at least one disc) over \mathcal{P} , with global basepoint O , and basepoint O_Δ for each disc Δ . (Note that since R is not periodic, there will be just one basepoint for each disc.) We will also choose, for each region R , a point O_R in the interior of R .

We can relabel \mathbb{P} to obtain a picture $\tilde{\mathbb{P}}$ over $\tilde{\mathcal{P}}$ as follows:

(a) For each region R , choose a tranverse path γ_R from O to O_R , and let U_R (a word on \mathbf{x}) be the label on the path γ_R . Then the *potential* $q(R)$ of R is $\theta(U_R)$. (This is independent of the choice of path γ_R , since $\theta(W) = 0$.)

(b) For an arc transversely labelled $x \in \mathbf{x}$ say, relabel it by $(q(R), x)$ where R is the region where the tranverse arrow on the arc begins.

(c) For a corner of a disc, with label $h \in H$ say, relabel the corner by (q, h) , where q is the potential of the region in which the corner occurs.

For a disc Δ , let q_Δ be the potential of the region containing O_Δ . Then in the relabelled picture, Δ will be labelled by the path (q_Δ, R) .

Let Θ be a *minimal* disc, that is, a disc such that $q_\Theta \leq q_\Delta$ for all discs Δ . Let m be the minimum value of ϕ_W^θ , and let e be one of the two distinct letters occurring at the unique minimum. Then in the path $(0, R)$ there is a unique edge labelled (m, e) . Now Θ is labelled by (q_Θ, R) in $\tilde{\mathbb{P}}$, and thus there is a unique edge labelled $(m + q_\Theta, e)$ incident with Θ . This arc must intersect another disc Θ' , which must also be labelled by (q_Θ, R) , but with the opposite orientation. Thus we obtain a dipole in $\tilde{\mathbb{P}}$ where Θ, Θ' are the discs

of the dipole. Reverting to \mathbb{P} , this dipole in $\tilde{\mathbb{P}}$ gives rise to a dipole in \mathbb{P} .

Acknowledgement. I thank the referee for his/her helpful comments.

References

- [1] R.B.J.T.Allenby and C.Y.Tang, Residual finiteness of certain 1-relator groups: extensions of results of Gilbert Baumslag, *Math. Proc. Camb. Phil. Soc.* **97** (1985), 225-230.
- [2] G.Baumslag, Residually finite one-relator groups, *Bull. Amer. Math. Soc.* **73** (1967), 618-620.
- [3] G.Baumslag, Free subgroups of certain one-relator groups defined by positive words, *Math. Proc. Camb. Phil. Soc.* **93** (1985), 247-251.
- [4] G.Baumslag and D.Solitar, Some two-generator one-relator non-Hopfian groups, *Bull. Amer. Math. Soc.* **68** (1962), 199-201.
- [5] G.Baumslag, A.Miasnikov and V.Shpilrain, Open problems in combinatorial and geometric group theory, <http://zebra.sci.ccny.cuny.edu/web/nygtc/problems/>
- [6] W.A.Bogley and S.J.Pride, Aspherical relative presentations, *Proc. Edin. Math. Soc.* **35** (1992), 1-39.
- [7] O.Bogopolski, A.Martino, O.Maslakova and E.Ventura, The conjugacy problem is solvable for free-by-cyclic groups, *Bull. London Math. Soc.* **38** (2006), 787-794.
- [8] K.S.Brown, Trees, valuations, and the Bieri-Neumann-Strebel invariant, *Invent. Math.* **90**, (1987), 479-504.
- [9] V.Egorov, The residual finiteness of certain one-relator groups, *Algebraic Systems, Ivanov. Gos.Univ., Ivanovo* (1981), 100-121.
- [10] J.Howie and S.J.Pride, A spelling theorem for staggered generalized 2-complexes, with applications, *Invent. Math.* **76** (1984), 55-74.
- [11] Kourovka Notebook **15** (2002).
- [12] W.Magnus, A.Karrass and D.Solitar, *Combinatorial Group Theory* (Second Edition), Dover, New York, 1976.
- [13] J.Meier, Geometric invariants for Artin groups, *Proc. London Math. Soc.(3)* **74** (1997), 151-173.
- [14] C.F.Miller III, *On group-theoretic decision problems and their classification*, Annals Of Mathematics Studies **68**, Princeton University Press, 1971.
- [15] S.Meskin, Nonresidually finite one-relator groups, *Trans. Amer. Math. Soc.* **164** (1972), 105-114.
- [16] B.B. Newman, Some results on one-relator groups, *Bull. Amer. Math. Soc.* **74** (1968), 568-571.
- [17] S.J.Pride, Star-complexes, and the dependence problems for hyperbolic complexes, *Glasgow Math. J.* **30** (1988), 155-170.
- [18] J.-P.Serre, *Trees*, Springer-Verlag, Berlin Heidelberg New York, 1980.
- [19] D.Wise, The residual finiteness of positive one-relator groups, *Comment. Math. Helv.* **76** (2001), 314-338.
- [20] D.Wise, Residual finiteness of quasi-positive one-relator groups, *J. London Math. Soc.(2)* **66** (2002), 334-350.

Address:

Department of Mathematics, University of Glasgow, University Gardens, Glasgow G12
8QW, Scotland UK

sjp@maths.gla.ac.uk